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Connection coefficients between Boas–Buck polynomial sets

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Abstract

In this paper, a general method to express explicitly connection coefficients between two Boas–Buck polynomial sets is presented. As application, we consider some generalized hypergeometric polynomials, from which we derive some well-known results including duplication and inversion formulas.

Keywords: Connection coefficients; Inversion formulas; Boas–Buck polynomials; Lowering operator; Generalized hypergeometric polynomials; Lagrange's expansion; Humbert polynomials; Panda polynomials

1. Introduction

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$. A polynomial sequence $\{P_n\}_{n\geqslant 0}$ in $\mathcal{P}$ is called a *polynomial set* if and only if $\deg P_n = n$.

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Given two polynomial sets $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, the so-called *connection problem* between them asks to find the coefficients $C_m(n)$ in the expression:

$$Q_n(x)=\sum_{m=0}^{n} C_m(n)P_m(x). \tag{1.1}$$

For the particular case $Q_n(x)=x^n$ the connection problem (1.1) is called *inversion problem* associated to $\{P_n(x)\}_{n\geqslant 0}$. In case of classical discrete orthogonal polynomials (Charlier, Meixner, Krawtchouk and Hahn) the falling factorial basis $x^{[n]}=(-1)^n(-x)_n$ ($(x)_n := \Gamma(x+n)/\Gamma(x)$ being the well-known Pochhammer's symbol) is more natural than x^n.

The connection coefficients play an important role in many problems in pure and applied mathematics or in mathematical physics. In fact, some inversion problems have been solved as one of the steps leading to the orthogonality for the corresponding polynomial sets. Moreover, the use of inversion in order to solve connection problems was considered by Rainville [18] (Hermite, Laguerre and Legendre polynomials) and by Gasper [9] (classical discrete orthogonal polynomials). The literature on this topic is extremely vast and a wide variety of methods, based on specific properties of the involved polynomials, have been devised for computing the connection coefficients. A known expansion of hypergeometric functions in hypergeometric functions called Fields and Wimp formula is firstly used by Lewanowicz [13] to find some connection coefficients when the involved polynomials are classical orthogonal polynomials of a discrete and continuous variables. The same formula was used by Sánchez-Ruiz and Dehesa [27] to solve connection problems for hypergeometric polynomials in and beyond the Askey scheme [12]. A recurrent approach called `navima`-algorithm which consists a method in computing $C_m(n)$ as solution of a recurrence relation can also be applied to solve connection and linearization problems (see, for instance, [19–25]). A general method which does not need particular properties of the polynomials involved in the problem based on the corresponding lowering operators and dual sequences [1] was developed in [3] to express explicitly the connection coefficients between two given polynomial sets.

In this paper, we deeply discuss the case where the involved polynomial sets have generating functions of Boas–Buck type (see Definition 2.1). Then we apply the obtained results to some classes of generalized hypergeometric polynomials as

$${}_{p+1}F_q\left(\begin{matrix}-n,\ (a_p)\\ (b_q)\end{matrix};x\right), \tag{1.2}$$

$${}_{p+2}F_q\left(\begin{matrix}-n,\ \lambda+n,\ (a_p)\\ (b_q)\end{matrix};x\right), \tag{1.3}$$

$${}_{p+1}F_{q+1}\left(\begin{matrix}-n,\ (a_p)\\ \alpha+n\beta+1,\ (b_q)\end{matrix};x\right), \tag{1.4}$$

and

$${}_{p+r}F_{q+r}\left(\begin{matrix}-n,\ \Delta(r-1,\lambda+n),\ (a_p)\\ \Delta(r,\lambda),\ (b_q)\end{matrix};(r-1)^{r-1}x\right), \tag{1.5}$$

where the ${}_pF_q$, as usual, denotes the generalized hypergeometric functions with p numerator and q denominator parameters. The contracted notation (a_p) is used to abbreviate the array of p parameters $a_1, \dots, a_p$, the parameters $r, \lambda, \alpha, \beta, (a_p), (b_q) \dots$ are assumed to be independent of n and $\Delta(r;\lambda)$ abbreviates the array of r parameters $(\lambda + j - 1)/r$, $j = 1, \dots, r$.

Many well-known polynomial sets and their various generalizations throughout the literature have hypergeometric representations of the types (1.2)–(1.5).

The polynomials defined by (1.5) are known as Jain polynomials [30, p. 179]. They include as particular case Srivastava–Pathan polynomials defined by (1.3) containing Sister Celine's $f_n((a_p), (b_q); x)$ [18, p. 290], Cohen's $\Theta(\nu, (a_p), (b_q); x)$ [27], generalized Rice's $H_n^{(\alpha,\beta)}(\varsigma, p, x)$, shifted Jacobi's $P_n^{(\alpha,\beta)}(x-1)$, generalized Bessel's $y_n(x, \alpha, \beta)$, and shifted Jacobi–Sobolev polynomials.

The structure of the paper is as follows. In Section 2, we prove our main result Theorem 2.5 and some useful corollaries. In Section 3, we apply the obtained results to many generalized hypergeometric polynomial sets. Finally, in Section 4, the explicit connection and inversion coefficients between classical orthogonal polynomials (continuous and discrete) are summarized in Tables 1–4 (see Section 4).

2. Connection coefficients between Boas–Buck polynomial sets

2.1. Preliminaries

Let us introduce, first, the following definitions and results. We will denote by:

- $\Lambda^{(-1)}$ the space of operators σ, acting on analytic functions, that reduce the degree of every polynomial by exactly one and $\sigma(1) = 0$;
- $\mathcal{S}_0$ the set of all power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_0 \neq 0$;
- $\hat{\mathcal{S}}_0$ the set of all power series $B(t) = \sum_{n=0}^{\infty} b_n t^n$, $b_n \neq 0$ for all n;
- $\mathcal{S}_1$ the set of all power series $C(t) = \sum_{n=1}^{\infty} c_n t^n$, $c_1 \neq 0$.

Let $C \in \mathcal{S}_1$. By C^* we mean the composition inverse of C, i.e., $C^*(C(t)) = t$, and $C(C^*(s)) = s$, $C^* \in \mathcal{S}_1$.

Let $\gamma := \{\gamma_n\}_{n\geqslant 0}$ be a complex numbers sequence satisfying $\gamma_n \neq 0$ for all nonnegative integer n. $\mathcal{D}_\gamma$ designates the operator in $\Lambda^{(-1)}$ defined by

$$\mathcal{D}_\gamma(1) = 0 \quad \text{and} \quad \mathcal{D}_\gamma(x^n) = \frac{\gamma_{n-1}}{\gamma_n} x^{n-1}, \quad n = 1, 2, \dots. \tag{2.1}$$

Definition 2.1. Let $\{P_n\}_{n\geqslant 0}$ be a polynomial set. $\{P_n\}_{n\geqslant 0}$ is said to have a generating function of Boas–Buck type (or a Boas–Buck polynomial set) if there exists a sequence of nonzero numbers $(\lambda_n)_{n\geqslant 0}$ such that

$$\sum_{n=0}^{\infty} \lambda_n P_n(x) t^n = A(t) B\bigl(xC(t)\bigr), \quad \text{where } (A, B, C) \in \mathcal{S}_0 \times \hat{\mathcal{S}}_0 \times \mathcal{S}_1. \tag{2.2}$$

The choice of $C(t) = t$ gives the class of Brenke polynomials.

In [5, p. 18], the authors referred to polynomials generated by (2.2) as generalized Appell polynomials but we do not do the same in this definition in order to avoid confusion with other extended Appell polynomials in the literature as, for instance, the polynomials introduced by Osegov [15].

Definition 2.2. Let $\sigma \in \Lambda^{(-1)}$ and $\{P_n\}_{n\geqslant 0}$ be a polynomial set. $\{P_n\}_{n\geqslant 0}$ is called a σ-Appell polynomial set if and only if

$$\sigma(P_n) = nP_{n-1}, \quad n = 1, 2, \dots .$$

σ is called lowering operator associated to $\{P_n\}_{n\geqslant 0}$.

In [3], the connection problem between polynomial sets was solved by the use of a method depending on commutativity of the lowering operators corresponding to the involved polynomials. In this paper, we deeply discuss the case where the polynomials, considered in (1.1), have generating functions of Boas–Buck type. So we have to recall the following lemma giving the lowering operator associated to a Boas–Buck polynomial set.

Lemma 2.3. (Cf. [2, Corollary 3.3]) *Let $\{P_n\}_{n\geqslant 0}$ be a Boas–Buck polynomial set generated by*

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n = A(t)B\big(xC(t)\big), \quad \text{where } (A, B, C) \in \mathcal{S}_0 \times \hat{\mathcal{S}}_0 \times \mathcal{S}_1.$$

Then $C^*(\mathcal{D}_\gamma)$ is the lowering operator associated to $\{P_n\}_{n\geqslant 0}$, where $\gamma := \{\gamma_n\}_{n\geqslant 0}$ and $B(t) = \sum_{n=0}^{\infty} \gamma_n t^n$.

The main relevance of this lemma is the fact that the lowering operator corresponding to any Boas–Buck polynomial set commutates with an operator of type $\mathcal{D}_\gamma$.

Sometimes, we need to replace P_n by $c_n P_n$, $c_n \neq 0$, in order to facilitate the derivation of the corresponding lowering operators. It is obvious to see that if the normalization is changed, say: $P_n = c_n \tilde{P}_n$, $Q_n = d_n \tilde{Q}_n$, then the new connection coefficients $\tilde{C}_m(n)$ are given by

$$\tilde{C}_m(n) = \frac{d_m}{c_n} C_m(n).$$

That means that we do not loss the generality of the problem if we limit ourselves to the case $\lambda_n = 1/n!$ in (2.2).

Notice finally that if $\{P_n\}_{n\geqslant 0}$ is a Boas–Buck polynomial set then $\{\hat{P}_n(x) := P_n(ax)\}_{n\geqslant 0}$, where a designates a nonzero complex number, is also a Boas–Buck polynomial set. So our treatment can be extended to the generalized connection problem

$$Q_n(ax) = \sum_{m=0}^{n} C_m(n, a) P_m(x). \tag{2.3}$$

2.2. Main result

Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be two polynomial sets of Boas–Buck type generated, respectively, by

$$A_1(t)B_1\big(xC_1(t)\big)=\sum_{n=0}^{\infty}\frac{P_n(x)}{n!}t^n \quad\text{and}\quad A_2(t)B_2\big(xC_2(t)\big)=\sum_{n=0}^{\infty}\frac{Q_n(x)}{n!}t^n, \tag{2.4}$$

where $(A_i,B_i,C_i)\in(\mathcal{S}_0\times\hat{\mathcal{S}}_0\times\mathcal{S}_1)$, $i=1,2$, and $B_i(t)=\sum_{k=0}^{\infty}\gamma_k^{(i)}t^k$, $i=1,2$.

It is easy to verify that the lowering operators associated respectively to $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ commutate if and only if $B_1=B_2$. For this case, the connection coefficients were derived as follows.

Lemma 2.4. (Cf. [3, Corollary 3.9]) *Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be two polynomial sets of Boas–Buck type given by* (2.4) *where $B_1=B_2$. Then the connection coefficients defined by* (1.1) *are given by*

$$\frac{A_2(t)}{A_1(\Psi^*(t))}\Psi^{*m}(t)=\sum_{n=0}^{\infty}\frac{m!}{n!}C_m(n)t^n, \tag{2.5}$$

or, equivalently,

$$C_m(n)=\frac{n!}{m!}\sum_{k=m}^{n}\psi_{n,k}^{*}\alpha_{k-m},$$

where

$$\Psi(t)=C_2^*\big(C_1(t)\big),\qquad \Psi^{*k}(t)=\sum_{r=k}^{\infty}\psi_{r,k}^{*}t^r,\quad\textit{and}\quad \frac{A_2(\Psi(t))}{A_1(t)}=\sum_{k=0}^{\infty}\alpha_k t^k.$$

Our purpose now is to generalize Lemma 2.4 by considering two Boas–Buck polynomial sets for which the corresponding lowering operators do not necessarily commutate. We state the following.

Theorem 2.5. *Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be two polynomial sets of Boas–Buck type given by* (2.4). *Then the connection coefficients defined by* (2.3) *are given by*

$$C_m(n,a)=\frac{n!}{m!}\sum_{k=m}^{n}a_k(n)b_m(k)a^k\frac{\gamma_k^{(2)}}{\gamma_k^{(1)}}, \tag{2.6}$$

where

$$A_2(t)C_2^m(t)=\sum_{n=m}^{\infty}a_m(n)t^n,\qquad \frac{C_1^{*m}(t)}{A_1(C_1^*(t))}=\sum_{n=m}^{\infty}b_m(n)t^n,\quad\textit{and}$$

$$B_i(t)=\sum_{k=0}^{\infty}\gamma_k^{(i)}t^k,\quad i=1,2.$$

Proof. Let $q_n(x) = n!a^n\gamma_n^{(2)}x^n$. Since $\{q_n\}_{n\geqslant 0}$ is a Boas–Buck polynomial set generated by $B_2(axt)$, using Lemma 2.4, the connection coefficients in

$$Q_n(ax) = \sum_{m=0}^{n} D_m(n)q_m(x)$$

are given by

$$A_2(t)C_2^m(t) = \sum_{n=m}^{\infty} \frac{m!}{n!} D_m(n)t^n.$$

Similar procedure leads to the fact that the connection coefficients in

$$p_n(x) = n!\gamma_n^{(1)}x^n = \sum_{m=0}^{n} I_m(n)P_m(x)$$

are given by

$$\frac{C_1^{*m}(t)}{A_1(C_1^*(t))} = \sum_{n=m}^{\infty} \frac{m!}{n!} I_m(n)t^n.$$

As the connection coefficients between the two basic sequences $\{q_n\}_{n\geqslant 0}$ and $\{p_n\}_{n\geqslant 0}$ are trivial, we obtain the desired expression (2.6) by composition. $\square$

It follows from this proof that the Boas–Buck polynomial set $\{P_n\}_{n\geqslant 0}$ has the following inversion formula:

$$\gamma_n^{(1)}x^n = \sum_{m=0}^{n} \frac{b_m(n)}{m!} P_m(x). \tag{2.7}$$

Next we give some useful consequences of Theorem 2.5.

2.3. Corollaries

Corollary 2.6 *(Connection between Brenke polynomial sets). Under the assumptions of Theorem* 2.5 *with $C_1(t) = C_2(t) = t$, the corresponding inversion and connection relations between two Brenke polynomial sets are given by*

$$\gamma_n^{(1)}x^n = \sum_{m=0}^{n} \frac{b_m(n)}{m!} P_m(x) \tag{2.8}$$

and

$$Q_n(ax) = \sum_{m=0}^{n} \frac{n!}{m!} \sum_{k=m}^{n} a_k(n)b_m(k)a^k \frac{\gamma_k^{(2)}}{\gamma_k^{(1)}} P_m(x), \tag{2.9}$$

where

$$A_2(t)t^m = \sum_{n=m}^{\infty} a_m(n)t^n \quad \text{and} \quad \frac{t^m}{A_1(t)} = \sum_{n=m}^{\infty} b_m(n)t^n.$$

Two particular cases are worthy to note:

Case 1. $A_1(t) = A_2(t) = e^t$. For this case, we have the inversion formula

$$\gamma_n^{(1)} x^n = \sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m)!m!} P_m(x) \tag{2.10}$$

and the generalized connection formula:

$$Q_n(ax) = \sum_{m=0}^{n} \binom{n}{m} a^m \sum_{k=0}^{n-m} (m-n)_k \frac{\gamma_{k+m}^{(1)}}{\gamma_{k+m}^{(2)}} \frac{a^k}{k!} P_m(x). \tag{2.11}$$

Case 2. $A_1(t) = (1-t)^{-\lambda}$ and $A_2(t) = (1-t)^{-\mu}$. For this case, we have

$$\gamma_n^{(1)} x^n = \sum_{m=0}^{n} \frac{(-\lambda)_{n-m}}{(n-m)!m!} P_m(x) \tag{2.12}$$

and

$$Q_n(ax) = \sum_{m=0}^{n} \binom{n}{m} (\mu)_{n-m} a^m \sum_{k=0}^{n-m} \frac{(m-n)_k(-\lambda)_k}{(1-\mu-n+m)_k} \frac{\gamma_{k+m}^{(2)}}{\gamma_{k+m}^{(1)}} \frac{a^k}{k!} P_m(x). \tag{2.13}$$

By virtue of Chu–Vandermonde formula [30, p. 30]:

$${}_2F_1\left(\begin{matrix} -n,\ b \\ c \end{matrix}; 1 \right) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, \ldots \text{ and } c \neq 0, -1, -2 \ldots, \tag{2.14}$$

the identity (2.13) with $B_1 = B_2$ and $a = 1$ is reduced to

$$Q_n(x) = \sum_{m=0}^{n} \binom{n}{m} (\lambda - \mu)_{n-m} P_m(x). \tag{2.15}$$

Corollary 2.7. *Let $\{Q_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ be two polynomial sets generated respectively by*

$$\sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} t^n = \big(1 + \xi_{\beta_2}(t)\big)^{\alpha_2} B_2\big(x\xi_{\beta_2}(t)\big) \quad \text{and}$$
$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n = \big(1 + \xi_{\beta_1}(t)\big)^{\alpha_1} B_1\big(x\xi_{\beta_1}(t)\big),$$

where α_i, β_i are any complex numbers and ξ_{β_i}, $i = 1, 2$, is a function of t defined implicitly by

$$\xi_{\beta_i} = t(1 + \xi_{\beta_i})^{\beta_i + 1}, \quad \xi_{\beta_i}(0) = 0.$$

Then the solutions of the inversion and generalized connection problems (2.3) *are given by*

$$n!\gamma_n^{(1)} x^n = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \big(m(\beta_1 + 1) + \alpha_1\big)_{n-m} P_m(x) \tag{2.16}$$

and

$$Q_n(ax)=\sum_{m=0}^{n}\binom{n}{m}\frac{(\alpha_2+n\beta_2)_n a^m}{(\alpha_2+n\beta_2)_{m+1}}\sum_{k=0}^{n-m}\frac{(m-n)_k}{k!}$$
$$\times\frac{(m(\beta_1+1)+\alpha_1)_k(\alpha_2+(\beta_2+1)(m+k))}{(\alpha_2+n\beta_2+m+1)_k}\frac{\gamma_{k+m}^{(2)}}{\gamma_{k+m}^{(1)}}a^k P_m(x). \tag{2.17}$$

We need the following lemma ([33, p. 133], [30, p. 354]) to prove this corollary.

Lemma 2.8 *(Lagrange's expansion). Let $f(z)$ and $\phi(z)$ be two functions of z analytic about the origin such that $\phi(0)\neq 0$, and let $z=w\phi(z)$. Then*

$$f(z)=f(0)+\sum_{n=1}^{\infty}\frac{w^n}{n!}D^{n-1}\left\{f'(z)\left[\phi(z)\right]^n\right\}_{z=0}, \tag{2.18}$$

where $D=\frac{d}{dz}$.

A useful consequence of (2.18) is ([17, p. 348], [30, p. 355]):

$$\left(1+\xi(t)\right)^{\alpha}=\sum_{n=0}^{\infty}\frac{\alpha}{\alpha+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}t^n, \tag{2.19}$$

where α and β are complex numbers independent of n and ξ is a function of t implicitly defined by

$$\xi=t(1+\xi)^{\beta+1},\quad \xi(0)=0. \tag{2.20}$$

In order to derive (2.19) from (2.18), we set $f(z)=(1+z)^{\alpha}$, $\phi(z)=(1+z)^{\beta+1}$ and replace respectively w and z by t and ξ.

Proof of Corollary 2.7. For this case, we have

$$C_i(t)=\xi_{\beta_i}(t)\quad\text{and}\quad A_i(t)=\left(1+\xi_{\beta_i}(t)\right)^{\alpha_i},\quad i=1,2.$$

The expansion (2.19) with $\xi(t)=C_2(t)$, $\beta=\beta_2$ and $\alpha=(\beta_2+1)m+\alpha_2$ gives

$$\begin{aligned}A_2(t)C_2^m(t)&=t^m\left(1+C_2(t)\right)^{(\beta_2+1)m+\alpha_2}\\&=\sum_{n=m}^{\infty}\frac{\alpha_2+m(\beta_2+1)}{\alpha_2+n(\beta_2+1)}\frac{(\alpha_2+n\beta_2+m+1)_{n-m}}{(n-m)!}t^n.\end{aligned}$$

On the other hand, since $C_1^*(t)=t/(1+t)^{\beta_1+1}$, we have

$$\frac{1}{A_1(C_1^*(t))}C_1^{*m}(t)=\frac{t^m}{(1+t)^{m\beta_1+m+\alpha_1}}=\sum_{n=m}^{\infty}\frac{(m(\beta_1+1)+\alpha_1)_{n-m}}{(n-m)!}(-1)^{n-m}t^n.$$

Then by virtue of (2.6), we derive (2.16) and (2.17). □

A similar proof may be used to prove the following.

Corollary 2.9 *(Connection between Panda polynomial sets). The Panda polynomials are generated by* [16]

$$\sum_{n=0}^{\infty} g_n^{\lambda}(x,r)t^n = (1-t)^{-\lambda} B_1\big(xt(1-t)^{-r}\big), \tag{2.21}$$

where $B_1(z)=\sum_{n=0}^{\infty}\gamma_n^{(1)}z^n$, $\gamma_n^{(1)}\neq 0$ *for* $n=0,1,\ldots$.

The corresponding inversion and connection formulas are given by

$$\frac{\gamma_n^{(1)}}{(\lambda)_{rn}}x^n = \sum_{m=0}^{n}\frac{(-1)^{n-m}}{(n-m)!}\frac{mr+\lambda}{(\lambda)_{n(r-1)+m+1}}g_m^{\lambda}(x,r) \tag{2.22}$$

and

$$g_n^{\mu}(ax,s) = \sum_{m=0}^{n}\frac{(sm+\mu)_{n-m}}{(n-m)!}\sum_{k=0}^{n-m}\frac{(m-n)_k}{k!}\frac{(n+m(s-1)+\mu)_{k(s-1)}(mr+\lambda)_{rk}}{(sm+\mu)_{sk}(mr+\lambda+1)_{k(r-1)}} \times \frac{\gamma_{k+m}^{(2)}}{\gamma_{k+m}^{(1)}}a^{k+m}g_m^{\lambda}(x,r). \tag{2.23}$$

The modified Panda polynomials $\{g_n^{\lambda-n}(x,r)\}_{n\geqslant 0}$, *generated by* [30, p. 178]

$$\sum_{n=0}^{\infty} g_n^{\lambda-n}(x,r)t^n = (1+t)^{\lambda-1} B_1\big(xt(1+t)^{r-1}\big), \tag{2.24}$$

have the inversion formula

$$\gamma_n^{(1)}(\lambda)_{(r-1)n}x^n = \sum_{m=0}^{n}(-1)^{m+1}\,\frac{m(1-r)+1-\lambda}{(n-m)!}(\lambda)_{n(r-1)-m-1}g_m^{\lambda-m}(x,r), \tag{2.25}$$

and the connection formula

$$g_n^{\mu-n}(ax,s) = \sum_{m=0}^{n}(-1)^{n-m}\frac{((1-s)m+(1-\mu))_{n-m}}{(n-m)!} \times \sum_{k=0}^{n-m}\frac{(m-n)_k}{k!}\frac{(\mu+(s-1)m)_{(s-1)k}(\lambda+m(r-1)+1)_{kr}}{((ms-n)-\mu+1)_{ks}(\lambda-mr)_{(r-1)k}}\frac{\gamma_{k+m}^{(2)}}{\gamma_{k+m}^{(1)}} \times a^{k+m}g_m^{\lambda-m}(x,r). \tag{2.26}$$

Proof. For this case, we have, $C_1(t) = -t/(1-t)^r$, $C_2(t) = -t/(1-t)^s$, $A_1(t) = (1-t)^{-\lambda}$ and $A_2(t) = (1-t)^{-\mu}$. Putting $\xi(t) = -C_1^*(t)$, $\beta = r-1$ and $\alpha = mr+\lambda$ in (2.19) we obtain

$$\frac{1}{A_1(C_1^*(t))}C_1^{*m}(t) = (-1)^m t^m \bigl(1 - C_1^*(t)\bigr)^{mr+\lambda} = \sum_{n=m}^{\infty} \frac{(-1)^m (mr+\lambda)(\lambda)_{rn}}{(n-m)!(\lambda)_{n(r-1)+m+1}} t^n. \tag{2.27}$$

On the other hand, it is easy to check that

$$A_2(t)C_2^m(t) = \sum_{n=m}^{\infty} (-1)^m \frac{(ms+\mu)_{n-m}}{(n-m)!} t^n. \tag{2.28}$$

Then, (2.23) follows from the identities (2.27)–(2.28) and Theorem 2.5. Using (2.7) and (2.27) we obtain (2.22). Finally, to derive (2.25) and (2.26), we use (2.22), (2.23), and the identity

$$g_n^{\lambda-n}(x,r) = (-1)^n g_n^{1-\lambda}(-x, 1-r),$$

which may be deduced from (2.21) and (2.24). □

An interesting special case of Corollary 2.9 is given by the following.

Corollary 2.10. *Let $\{Q_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ be two polynomial sets generated respectively by*

$$\sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} t^n = (1-t)^{-\mu} B_2\left(x \frac{-4t}{(1-t)^2}\right) \quad \text{and}$$
$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n = (1-t)^{-\lambda} B_1\left(x \frac{-4t}{(1-t)^2}\right).$$

Then, the expansions (2.22) *and* (2.23) *become*

$$\frac{n!4^n\gamma_n^{(1)}}{(\lambda)_{2n}} x^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(2m+\lambda)}{(\lambda)_{n+m+1}} P_m(x) \tag{2.29}$$

and

$$Q_n(ax) = \sum_{m=0}^{n} \binom{n}{m} (2m+\mu)_{n-m} \times \sum_{k=0}^{n-m} \frac{(m-n)_k}{k!} \frac{(m+n+\mu)_k(2m+\lambda)_{2k}}{(2m+\mu)_{2k}(2m+\lambda+1)_k} \frac{\gamma_{k+m}^{(2)}}{\gamma_{k+m}^{(1)}} a^{k+m} P_m(x). \tag{2.30}$$

Another consequence of Theorem 2.5 and Lagrange's expansion is given by the following.

Corollary 2.11 *(Connection between Humbert polynomial sets). The inversion and connection formulas for Humbert polynomial set* $\{h^{\nu}_{n,m}\}_{n\geqslant 0}$, *generated by* [30, p. 86]

$$\sum_{n=0}^{+\infty} h^{\nu}_{n,m}(x)t^n = (1 - mxt + t^m)^{-\nu}, \quad m = 1, 2, \ldots,$$

are

$$\frac{m^n}{n!}x^n = \sum_{k=0}^{[n/m]} \frac{\nu + n - mk}{(\nu)_{n+1-k}} \frac{h^{\nu}_{n-mk,m}(x)}{k!} \tag{2.31}$$

and

$$h^{\lambda}_{n,m}(ax) = \sum_{k=0}^{[n/m]} \frac{(\nu + n - mk)(\lambda)_n a^n}{k!(\nu)_{n+1-k}} \, {}_mF_{m-1}\left(\begin{matrix} -k, \Delta(m-1, k-\nu-n) \\ \Delta(m-1, 1-\lambda-n) \end{matrix}; a^{-m}\right)$$
$$\times h^{\nu}_{n-mk,m}(x). \tag{2.32}$$

Proof. For this case, we have

$$C_1(t) = C_2(t) = \frac{t}{1+t^m}, \qquad A_1(t) = (1+t^m)^{-\nu}, \qquad A_2(t) = (1+t^m)^{-\lambda},$$
$$B_1(t) = (1-mt)^{-\lambda}, \quad \text{and} \quad B_2(t) = (1-mt)^{-\mu}.$$

Applying (2.18) with $f(z) = (1 + z^m)^{\alpha}$ and $\phi(z) = 1 + z^2$, we obtain

$$(1+z^m)^{\alpha} = \sum_{k=0}^{+\infty} \frac{\alpha(k(m-1)+\alpha+1)_k}{(mk+\alpha)k!} w^{mk}. \tag{2.33}$$

Replacing in (2.33), w by t, z by $C_1^*(t)$, and α by $\nu + n$, we derive

$$\frac{1}{A_1(C_1^*(t))} C_1^{*n}(t) = t^n\bigl(1 + C_1^{*m}(t)\bigr)^{\nu+n}$$
$$= \sum_{k=0}^{\infty} \frac{(\nu+n)(k(m-1)+\nu+n+1)_k}{(mk+\nu+n)k!} t^{mk+n}. \tag{2.34}$$

On the other hand, it is obvious that

$$A_2(t)C_2^n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(n+\lambda)_k t^{n+mk}. \tag{2.35}$$

Then (2.32) follows from (2.34), (2.35), and Theorem 2.5.

Using (2.34) and (2.7) we obtain (2.31). □

The Humbert polynomials $\{h^{\nu}_{n,m}\}_{n\geqslant 0}$ contain as particular case, the Legendre ($m = 2$, $\nu = 1/2$), Tchebychev ($m = 2$, $\nu = 1$), Pincherle ($m = 3$, $\nu = -1/2$), Kinney ($m = r$, $\nu = 1/r$), Byrd ($m = 2$, $\nu = 1$) and Gegenbauer ($m = 2$, $\nu = \lambda$) polynomials. The inversion and connection formulas for Gegenbauer ones are, respectively, given in Tables 1 and 4 (see Section 4).

3. Applications

In this section, we use Theorem 2.5 and its consequences to derive the inversion and connection coefficients for some well-known polynomial sets.

3.1. Brafman polynomials

The Brafman polynomials $\mathcal{B}_n^1((a_p),(b_q);x)$ defined by the hypergeometric representation (1.2) are generated by [6,8]

$$\sum_{n=0}^{\infty}\mathcal{B}_n^1\big((a_p),(b_q);x\big)\frac{t^n}{n!}=e^t\,{}_pF_q\left(\begin{matrix}(a_p)\\(b_q)\end{matrix};-xt\right).$$

By virtue of (2.10), the corresponding inversion formula is

$$\frac{[a_p]_n}{[b_q]_n}x^n=\sum_{m=0}^{n}\binom{n}{m}(-1)^m\mathcal{B}_m^1\big((a_p),(b_q);x\big),\tag{3.1}$$

where $[a_p]_n=\prod_{i=1}^{p}(a_i)_n$.

Formula (3.1) will be applied to get the inversion coefficients for Laguerre, Krawtchouk, Meixner and Charlier polynomials (see Tables 1, 2, Section 4).

Using (2.11) we obtain the connection between two Brafman polynomial sets:

$$\begin{aligned}\mathcal{B}_n^1\big((c_r),(d_s);ax\big)=\sum_{m=0}^{n}\binom{n}{m}\frac{[c_r]_m[b_q]_m}{[d_s]_m[a_p]_m}a^m&\\ \times\,{}_{r+q+1}F_{s+p}\left(\begin{matrix}m-n,(c_r+m),(b_q+m)\\(d_s+m),(a_p+m)\end{matrix};a\right)&\\ \times\,\mathcal{B}_m^1\big((a_p),(b_q);x\big).&\end{aligned}\tag{3.2}$$

3.1.1. Laguerre polynomials

For the Laguerre polynomials

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\\alpha+1\end{matrix};x\right),$$

we obtain the following generalized multiplication formula:

$$L_n^{(\beta)}(ax)=\sum_{m=0}^{n}\frac{(\beta+1)_n}{(\beta+1)_m}\frac{a^m}{(n-m)!}\,{}_2F_1\left(\begin{matrix}n-m,\ \alpha+m+1\\\beta+m+1\end{matrix};a\right)L_m^{(\alpha)}(x).\tag{3.3}$$

For $a=1$, we obtain, by virtue of (2.14), the well-known connection formula between Laguerre polynomials (see Table 4). For the special case $\alpha=\beta$, (3.3) becomes [18, p. 209]

$$L_n^{(\alpha)}(ax)=\sum_{m=0}^{n}(\alpha+m+1)_{n-m}a^m(1-a)^{n-m}L_m^{(\alpha)}(x).$$

3.1.2. Koekoek–Laguerre–Sobolev polynomials

The Koekoek–Laguerre–Sobolev polynomials are defined by [11]

$$L_n^{\alpha,M_0,\dots,M_N}(x)=\frac{\beta_0\dots\beta_N}{(\alpha+1)_{N+1}}\frac{(\alpha+1)_n}{n!}(A_0+\cdots+A_{N+1})\\ \times {}_{N+2}F_{N+2}\left(\begin{matrix}-n,\ \beta_0+1,\ (\beta_N+1)\\ \alpha+N+2,\ \beta_0,\ (\beta_N)\end{matrix};x\right).$$

The corresponding connection formula is

$$\begin{aligned}&L_n^{\gamma,T_0,\dots,T_S}(ax)\\ &\quad=\sum_{m=0}^{n}\frac{(\alpha+1)_{N+1}}{(\gamma+1)_{S+1}}\frac{\lambda_0\dots\lambda_S}{\beta_0\dots\beta_N}\frac{(B_0+\cdots+B_{S+1})(\gamma+1)_n}{(A_0+\cdots+A_{N+1})(\alpha+1)_m}\frac{(\alpha+N+2)_m}{(\gamma+S+2)_m}\\ &\qquad\times {}_{N+S+4}F_{N+S+3}\left(\begin{matrix}m-n,\alpha+N+m+2,\lambda_0+m+1,(\lambda_S+m+1),\ \beta_0+m,(\beta_N+m)\\ \gamma+S+m+2,\ \lambda_0+m,\ (\lambda_S+m),\ \beta_0+m+1,\ (\beta_N+m+1)\end{matrix};a\right)\\ &\qquad\times\frac{a^m}{(n-m)!}\frac{\prod_{i=0}^{S}(\lambda_i+1)_m\prod_{i=0}^{N}(\beta_i)_m}{\prod_{i=0}^{S}(\lambda_i)_m\prod_{i=0}^{N}(\beta_i+1)_m}L_m^{\alpha,M_0,\dots,M_N}(x).\end{aligned}\tag{3.4}$$

The connection formula (3.4) can be obtained by use of Fields and Wimp formula [27].

The d-orthogonal polynomials studied by Ben Cheikh and Douak [4]:

$$L_n^{\alpha_1,\alpha_2,\dots,\alpha_d}(x)={}_1F_d\left(\begin{matrix}-n\\ (\alpha_d+1)\end{matrix};x\right),$$

have the expansions formulas

$$\frac{x^n}{[1+\alpha_d]_n}=\sum_{m=0}^{n}\binom{n}{m}(-1)^m L_m^{\alpha_1,\alpha_2,\dots,\alpha_d}(x)$$

and

$$L_n^{\alpha_1,\alpha_2,\dots,\alpha_d}(ax)=\sum_{m=0}^{n}\binom{n}{m}\frac{[\beta_e+1]_m}{[\alpha_d+1]_m}a^m\,{}_{1+e}F_d\left(\begin{matrix}m-n,(1+\beta_e+m)\\ (\alpha_d+1+m)\end{matrix};a\right)\\ \times L_m^{\beta_1,\beta_2,\dots,\beta_e}(x).$$

3.2. Chaunday polynomials

By (2.12) and (2.13), the hypergeometric polynomial set generated by [8]

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{p+1}F_{q+1}\left(\begin{matrix}-n,(a_p)\\ 1-\lambda-n,(b_q)\end{matrix};x\right)t^n=(1-t)^{-\lambda}{}_pF_q\left(\begin{matrix}(a_p)\\ (b_q)\end{matrix};xt\right),$$

have respectively the inversion formula

$$\frac{[a_p]_n}{[b_q]_n}x^n=\sum_{m=0}^{n}\binom{n}{m}(-\lambda)_{n-m}(\lambda)_m\,{}_{p+1}F_{q+1}\left(\begin{matrix}-m,(a_p)\\ 1-\lambda-m,(b_q)\end{matrix};x\right),$$

and the connection formula

$$
\begin{aligned}
&{}_{r+1}F_{s+1}\left(\begin{matrix}-n,(c_r)\\1-\mu-n,(d_s)\end{matrix};ax\right)\\
&\quad=\sum_{m=0}^{n}\binom{n}{m}\frac{a^m(\lambda)_m}{(\mu+n-m)_m}\frac{[c_r]_m[b_q]_m}{[d_s]_m[a_p]_m}\\
&\qquad\times{}_{r+q+2}F_{s+p+1}\left(\begin{matrix}m-n,\ -\lambda,\ (c_r+m),\ (b_q+m)\\1-\mu-n+m,\ (d_s+m),\ (a_p+m)\end{matrix};a\right)\\
&\qquad\times{}_{p+1}F_{q+1}\left(\begin{matrix}-m,(a_p)\\1-\lambda-m,(b_q)\end{matrix};x\right).
\end{aligned}
\tag{3.5}
$$

As application, the modified Laguerre polynomials $\{L_n^{(\gamma-n)}\}_{n\geqslant 0}$ have the expansion

$$
L_n^{(\gamma-n)}(x)=\sum_{m=0}^{n}\frac{(-1)^{n-m}}{(n-m)!}(\alpha-\gamma)_{n-m}L_m^{(\alpha-m)}(x).
\tag{3.6}
$$

3.3. *Jain polynomials*

The Jain polynomials $R_n^{(r)}(\lambda,(a_p),(b_q);x)$ [10,16], defined by (1.5) are generated by [30, p. 179]:

$$
\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}R_n^{(r)}\big(\lambda,(a_p),(b_q);x\big)t^n=(1-t)^{-\lambda}{}_pF_q\left(\begin{matrix}(a_p)\\(b_q)\end{matrix};-\frac{r^rxt}{(1-t)^r}\right).
$$

The inversion formula is

$$
\frac{[a_p]_n}{[b_q]_n}\frac{r^{nr}}{(\lambda)_{rn}}x^n=\sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{mr+\lambda}{(\lambda+m)_{n(r-1)+1}}R_m^{(r)}\big(\lambda,(a_p),(b_q);x\big).
\tag{3.7}
$$

By use of (2.23) we obtain the connection formula

$$
\begin{aligned}
&R_n^{(s)}\big(\mu,(c_t),(d_u);x\big)\\
&\quad=\sum_{m=0}^{n}\binom{n}{m}\frac{(\mu+sm)_{n-m}(\lambda)_m}{(\mu)_n}a^m\frac{[c_t]_m[b_q]_m}{[d_u]_m[a_p]_m}\frac{s^{ms}}{r^{rm}}\\
&\qquad\times{}_{t+r+s+q}F_{t+r+s+p-1}\left(\begin{matrix}m-n,\Delta_m(r,\lambda),\Delta_m(s-1,\mu+n),(c_t+m),(b_q+m)\\\Delta_m(s,\mu),\Delta(r-1,rm+\lambda+1),(a_p+m),(d_t+m)\end{matrix};\frac{(s-1)^{s-1}}{(r-1)^{r-1}}a^k\right)\\
&\qquad\times R_m^{(r)}\big(\lambda,(a_p),(b_q);x\big),
\end{aligned}
\tag{3.8}
$$

where $\Delta_m(r,\lambda)=\Delta(r,\lambda+rm),\ldots.$

To obtain (3.8) we have also used Gauss's multiplication theorem [30, p. 23]

$$
(\lambda)_{mn}=m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n.
$$

The connection formula (3.8) was given in [27] for the particular cases $r=2$ and $r=1$. The general case given here appears to be new.

3.4. Srivastava–Pathan polynomials

The Srivastava–Pathan polynomials defined by (1.3) are generated by [8]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{p+2}F_q\left(\begin{matrix}-n, \lambda+n, (a_p)\\ (b_q)\end{matrix}; x\right) t^n$$
$$= (1-t)^{-\lambda} {}_{p+2}F_q\left(\begin{matrix}\frac{\lambda}{2},\ \frac{\lambda+1}{2},\ (a_p)\\ (b_q)\end{matrix}; \frac{-4xt}{(1-t)^2}\right).$$

From (2.29), the following inversion formula can be easily derived

$$\frac{[a_p]_n}{[b_q]_n} x^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{2m+\lambda}{(\lambda+m)_{n+1}} {}_{p+2}F_q\left(\begin{matrix}-m, \lambda+m, (a_p)\\ (b_q)\end{matrix}; x\right). \tag{3.9}$$

By virtue of (2.30), we obtain the connection formula

$$\begin{aligned}
&{}_{r+2}F_s\left(\begin{matrix}-n, \mu+n, (c_r)\\ (d_s)\end{matrix}; ax\right)\\
&\quad = \sum_{m=0}^{n} \binom{n}{m} \frac{(\mu+n)_m}{(\lambda+m)_m} \frac{[c_r]_m [b_q]_m}{[d_s]_m [a_p]_m} a^m\\
&\qquad \times {}_{r+q+2}F_{s+p+1}\left(\begin{matrix}m-n,\ m+n+\mu,\ (c_r+m),\ (b_q+m)\\ 2m+\lambda+1,\ (a_p+m),\ (d_s+m)\end{matrix}; a\right)\\
&\qquad \times {}_{p+2}F_q\left(\begin{matrix}-m, \lambda+m, (a_p)\\ (b_q)\end{matrix}; x\right),
\end{aligned} \tag{3.10}$$

which is a particular case of (3.8).

The relation (3.9) will be applied to calculate inversion coefficients for Hahn and shifted Jacobi polynomials (Tables 1, 2). The expansion (3.10) solves connection problems for many well-known polynomial sets.

3.4.1. Wilson polynomials

The Wilson polynomials are defined by [12]

$$\begin{aligned}
W_n(x^2, a, b, c, d) &= (a+b)_n (a+c)_n (a+d)_n\\
&\quad \times {}_4F_3\left(\begin{matrix}-n,\ n+a+b+c+d-1,\ a+ix,\ a-ix\\ a+b,\ a+c,\ a+d\end{matrix}; 1\right).
\end{aligned}$$

The corresponding inversion-type formula is

$$\begin{aligned}
&\frac{(a+ix)_n (a-ix)_n}{(a+b)_n (a+c)_n (a+d)_n}\\
&\quad = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(2m+a+b+c+d-1)}{(a+b+c+d+m-1)_{n+1}} \frac{W_m(x^2, a, b, c, d)}{(a+b)_m (a+c)_m (a+d)_m}.
\end{aligned}$$

The connection between two Wilson polynomials with first common parameters is

$$\begin{aligned} &W_n(x^2,a,b,c,d)\\ &\quad=\sum_{m=0}^{n}\binom{n}{m}\frac{(a+b+c+d+n-1)_m(a+b)_n(a+c)_n(a+d)_n}{(a+e+f+g+m-1)_m(a+b)_m(a+c)_m(a+d)_m}\\ &\qquad\times {}_5F_4\left(\begin{matrix} m-n,\ m+n+a+b+c+d-1,\ a+e+m,\ a+f+m,\ a+g+m\\ 2m+a+e+f+g,\ a+b+m,\ a+c+m,\ a+d+m\end{matrix};1\right)\\ &\qquad\times W_m(x^2,a,e,f,g).\end{aligned}$$

In Table 3, we give all possible connection coefficients between two classical discrete orthogonal polynomials (Hahn, Krawtchouk, Meixner and Charlier) without using the orthogonality conditions. These coefficients can be obtained from Wilson's ones by taking limit relations or directly by (3.10).

3.4.2. Racah polynomials

The Racah polynomials are defined by [12]

$$R_n\big(\lambda(x),\alpha,\beta,\gamma,\delta\big)={}_4F_3\left(\begin{matrix}-n,\ n+\alpha+\beta+1,\ -x,\ x+\gamma+\delta+1\\ \alpha+1,\ \beta+\delta+1,\ \gamma+1\end{matrix};1\right),$$

where $\lambda(x)=x(x+\gamma+\delta+1)$.

The corresponding inversion formula is

$$\begin{aligned}&\frac{(-x)_n(x+\gamma+\delta+1)_n}{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}\\ &\quad=\sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{(2m+\alpha+\beta+1)}{(\alpha+\beta+m+1)_{n+1}}R_m\big(\lambda(x),\alpha,\beta,\gamma,\delta\big).\end{aligned}$$

If $d+c=\gamma+\delta$, we have

$$\begin{aligned} &R_n\big(\lambda(x),\alpha,\beta,\gamma,\delta\big)\\ &\quad=\sum_{m=0}^{n}\binom{n}{m}\frac{(\alpha+\beta+n+1)_m(a+1)_m(b+d+1)_m(c+1)_m}{(a+b+m+1)_m(\gamma+1)_m(\alpha+1)_m(\beta+\delta+1)_m}\\ &\qquad\times {}_5F_4\left(\begin{matrix} m-n,\ m+n+\alpha+\beta+1,\ a+m+1,\ b+d+m+1,\ c+m+1\\ 2m+a+b+2,\ \alpha+1,\ \beta+\delta+1,\ \gamma+1\end{matrix};1\right)\\ &\qquad\times R_m\big(\lambda(x),a,b,c,d\big).\end{aligned}$$

3.4.3. Jacobi polynomials

The connection coefficients for the Jacobi polynomials, defined by [18]

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\alpha+\beta+1+n\\ \alpha+1\end{matrix};\frac{1-x}{2}\right),$$

are given by

$$P_n^{(\gamma,\delta)}(x) = \sum_{m=0}^{n} \frac{1}{(n-m)!} \frac{(\gamma+1)_n(\mu+n)_m}{(\gamma+1)_m(\lambda+m)_m} \times {}_3F_2\left(\begin{matrix} m-n,\ m+n+\mu,\ \alpha+m+1 \\ \gamma+m+1,\ 2m+\lambda+1 \end{matrix}; 1\right) P_m^{(\alpha,\beta)}(x),$$

where $\lambda = \alpha+\beta+1$ and $\mu = \gamma+\delta+1$.

3.4.4. Generalized Rice polynomials

The generalized Rice polynomials are defined by [30, p. 140]

$$H_n^{(\alpha,\beta)}(\varsigma, p, x) = \frac{(\alpha+1)_n}{n!} {}_3F_2\left(\begin{matrix} -n,\ \lambda+n,\ \varsigma \\ \alpha+1,\ p \end{matrix}; x\right), \quad \text{with } \lambda = \alpha+\beta+1.$$

The corresponding inversion formula is

$$x^n = \sum_{m=0}^{n} (-n)_m \frac{(2m+\lambda)(\alpha+1)_n(p)_n}{(\lambda+m)_{n+1}(\varsigma)_n(\alpha+1)_m} H_m^{(\alpha,\beta)}(\varsigma, p, x).$$

The connection between two generalized Rice polynomial sets is

$$H_n^{(\gamma,\delta)}(\tau, q, ax) = \sum_{m=0}^{n} \frac{1}{(n-m)!} \frac{(\gamma+1)_n(\mu+n)_m(\tau)_m(p)_m}{(\alpha+1)_m(\lambda+m)_m(\varsigma)_m(q)_m} a^m \times {}_5F_4\left(\begin{matrix} m-n,\ m+n+\mu,\ \tau+m+1,\ p+m,\ \alpha+m+1 \\ \varsigma+m,\ 2m+\lambda+1,\ q+m,\ \gamma+m+1 \end{matrix}; a\right) \times H_m^{(\alpha,\beta)}(\varsigma, p, x),$$

where $\mu = \gamma+\delta+1$.

3.4.5. Generalized Bessel polynomials

The generalized Bessel polynomials [18, p. 294]

$$y_n(x, \alpha, \beta) = {}_2F_0\left(\begin{matrix} -n,\ \alpha+n-1 \\ - \end{matrix}; \frac{-x}{\beta}\right)$$

have the following inversion and connection formulas:

$$x^n = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \beta^n \frac{(2m+\alpha-1)}{(\alpha+m-1)_{n+1}} y_m(x, \alpha, \beta)$$

and

$$y_n(x, \gamma, \delta) = \sum_{m=0}^{n} \binom{n}{m} \frac{(\gamma+n-1)_m}{(\alpha-1+m)_m} \left(\frac{\beta}{\delta}\right)^m \times {}_2F_1\left(\begin{matrix} m-n,\ m+n+\gamma-1 \\ \gamma+m-1 \end{matrix}; \frac{\beta}{\delta}\right) y_m(x, \alpha, \beta).$$

By virtue of (2.14), the last connection formula makes a simple form if $\beta = \delta$:

$$y_n(x, \gamma, \beta) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \frac{(\gamma+n-1)_m(\gamma-\alpha)_{n-m}}{(\alpha-1+m)_m(2m+\alpha)_{n-m}} y_m(x, \alpha, \beta).$$

3.5. Srivastava polynomials

Srivastava introduced the polynomials [29]:

$$T_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(-n)_k}{(1+\alpha+n\beta)_k}\gamma_k^{(1)}x^k, \tag{3.11}$$

where α, β and $\gamma_k^{(1)}$ are arbitrary complex numbers such that $\gamma_k^{(1)}\neq 0$, $k=0,1,\ldots$.

A corresponding generating function is given by [31]

$$\sum_{n=0}^{\infty}\frac{\alpha}{\alpha+(\beta+1)n}\frac{(\alpha+n\beta+1)_n}{n!}T_n^{(\alpha,\beta)}(x)=(1+\xi)^{\alpha}B_1(-x\xi), \tag{3.12}$$

where ξ is defined by (2.20) and

$$B_1(z)=\sum_{k=0}^{\infty}\frac{\alpha}{\alpha+(\beta+1)k}\gamma_k^{(1)}z^k.$$

Using (2.16) and (2.17), we derive the inversion and connection formulas:

$$\frac{\gamma_n^{(1)}}{\alpha+(\beta+1)n}x^n=\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}(\alpha+m\beta+1)_{n-1}\,T_m^{(\alpha,\beta)}(x) \tag{3.13}$$

and

$$\begin{aligned}T_n^{(\gamma,\delta)}(ax)={}&\sum_{m=0}^{n}\binom{n}{m}\frac{(\alpha+m\beta+1)_m}{(\gamma+n\delta+1)_m}a^m\sum_{k=0}^{n-m}\frac{(m-n)_k}{k!}\frac{\alpha+(\beta+1)(m+k)}{\alpha+(\beta+1)m}\\&\times\frac{(m(\beta+1)+\alpha)_k}{(\gamma+n\delta+m+1)_k}\frac{\gamma_{k+m}^{(2)}}{\gamma_{k+m}^{(1)}}a^kT_m^{(\alpha,\beta)}(x).\end{aligned} \tag{3.14}$$

If we take

$$\gamma_k^{(1)}=\frac{[a_p]_k}{[b_q]_k k!}$$

in (3.11) and we use (3.12), we obtain the following generating function for the generalized hypergeometric polynomials defined in (1.4):

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\alpha}{\alpha+(\beta+1)n}(\alpha+n\beta+1)_n\,{}_{p+1}F_{q+1}\left(\begin{matrix}-n,\ (a_p)\\ \alpha+n\beta+1,\ (b_q)\end{matrix};x\right)\frac{t^n}{n!}\\&\quad=\big(1+\xi(t)\big)^{\alpha}\,{}_{p+1}F_{q+1}\left(\begin{matrix}\frac{\alpha}{\beta+1},\ (a_p)\\ 1+\frac{\alpha}{\beta+1},\ (b_q)\end{matrix};-x\xi(t)\right),\end{aligned}$$

where $\xi(t)$ is defined by (2.20). This identity was derived by Brown [7] for the special case $\beta=-\frac{1}{2}$ and generalized by Srivastava ([30, p. 365], [31]).

The inversion formula (3.13) is reduced to

$$\frac{[a_p]_n}{[b_q]_n}\frac{x^n}{\alpha+n(\beta+1)}=\sum_{m=0}^{n}(-1)^m\binom{n}{m}(m\beta+\alpha+1)_{n-1}\\ \times {}_{p+1}F_{q+1}\left(\begin{matrix}-m,\ (a_p)\\ \alpha+m\beta+1,\ (b_q)\end{matrix};x\right). \tag{3.15}$$

By use of (3.14), we obtain the connection formula

$${}_{r+1}F_{s+1}\left(\begin{matrix}-n,\ (c_r)\\ \gamma+n\delta+1,\ (d_s)\end{matrix};ax\right)\\ =\sum_{m=0}^{n}\binom{n}{m}\frac{(\alpha+m\beta+1)_m}{(\gamma+n\delta+1)_m}\frac{[c_r]_m[b_q]_m}{[d_s]_m[a_p]_m}a^m\\ \times {}_{r+q+3}F_{s+p+2}\left(\begin{matrix}m-n,\ 1+\frac{\alpha}{\beta+1}+m,\ m(\beta+1)+\alpha,\ (c_r+m),\ (b_q+m)\\ \frac{\alpha}{\beta+1}+m,\ \gamma+n\delta+m+1,\ (a_p+m),\ (d_s+m)\end{matrix};a\right)\\ \times {}_{p+1}F_{q+1}\left(\begin{matrix}-m,\ (a_p)\\ \alpha+m\beta+1,\ (b_q)\end{matrix};x\right). \tag{3.16}$$

Next, we consider two particular cases of (3.15) and (3.16).

3.5.1. *Modified Jacobi polynomials*

In the modified Jacobi polynomials case,

$$P_n^{(\alpha+bn,\ \beta-(b+1)n)}(x)=\frac{(\alpha+nb+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\alpha+\beta+1\\ \alpha+nb+1\end{matrix};\frac{1-x}{2}\right),$$

we find that

$$x^n=\sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{m!(mb+\alpha+1)_{n-1}(\alpha+n(b+1))}{(1+\alpha+\beta)_n(1+\alpha+mb)_m}\\ \times P_m^{(\alpha+bm,\ \beta-(b+1)m)}(1-2x)$$

and

$$P_n^{(\gamma+cn,\ \delta-(c+1)n)}(x)\\ =\sum_{m=0}^{n}\frac{1}{(n-m)!}\frac{(1+\gamma+nc)_n(1+\gamma+\delta)_m}{(1+\gamma+nc)_m(1+\alpha+\beta)_m}\\ \times {}_4F_3\left(\begin{matrix}m-n,\ \alpha+m(b+1),\ \gamma+\delta+1+m,\ 1+\frac{\alpha}{b+1}+m\\ \gamma+cn+m+1,\ \alpha+\beta+1+m,\ \frac{\alpha}{b+1}+m\end{matrix};1\right)\\ \times P_m^{(\alpha+bm,\ \beta-(b+1)m)}(x). \tag{3.17}$$

A particular simple form of (3.17) may be obtained if we take $\alpha+\beta=\gamma+\delta$ and we use the following reduction formula [14, p. 110]:

$$ {}_3F_2\left(\begin{matrix} -n,\ b,\ c+1 \\ d,\ c \end{matrix};1\right) = \frac{(d-b)_n}{(d)_n}\left(1+\frac{nb}{c(b+1-n-d)}\right). \tag{3.18} $$

We have, in fact,

$$ P_n^{(\gamma+cn,\ \delta-(c+1)n)}(x) = \sum_{m=0}^{n} \frac{(\gamma-\alpha+1+nc-mb)_{n-m}(\alpha-\gamma+n(b-c))}{(n-m)!(\alpha-\gamma+m(b+1)-n(c+1))} \times P_m^{(\alpha+bm,\ \beta-(b+1)m)}(x). $$

3.5.2. Modified Laguerre polynomials

For the modified Laguerre polynomials

$$ L_n^{(\alpha+n\beta)}(x) = \frac{(\alpha+n\beta+1)_n}{n!}\,{}_1F_1\left(\begin{matrix} -n \\ \alpha+n\beta+1 \end{matrix};x\right), $$

we state the inversion formula,

$$ x^n = \sum_{m=0}^{n}(-n)_m \frac{(m\beta+\alpha+1)_{n-1}(\alpha+n(\beta+1))}{(m\beta+\alpha+1)_m} L_m^{(\alpha+m\beta)}(x), \tag{3.19} $$

and the connection relation

$$ L_n^{(\gamma+\delta n)}(ax) = \sum_{m=0}^{n} \frac{a^m}{(n-m)!}\,{}_3F_2\left(\begin{matrix} m-n,\ m(\beta+1)+\alpha,\ 1+\frac{\alpha}{\beta+1}+m \\ \gamma+n\delta+m+1,\ \frac{\alpha}{\beta+1}+m \end{matrix};a\right) \times (n\delta+\gamma+m+1)_{n-m} L_m^{(\alpha+m\beta)}(x). \tag{3.20} $$

Again, by virtue of (3.18), with $a=1$, (3.20) is reduced to

$$ L_n^{(\gamma+\delta n)}(x) = \sum_{m=0}^{n} \frac{(\gamma-\alpha+n\delta-m\beta+1)_{n-m}(\alpha-\gamma+n(\beta-\delta))}{(n-m)!(\alpha-\gamma+m(\beta+1)-n(\delta+1))}\ L_m^{(\alpha+\beta m)}(x). \tag{3.21} $$

If we take $\beta=0$ in (3.19) or $\delta=\beta=0$ in (3.21) we rediscover some well-known formulas (see Tables 1 and 4, respectively).

Some of the above inversion and connection formulas have been previously found via different methods either analytically [9,26–28,34] or recurrently [13,20,21]. For some cases, the recurrence relation for the connection coefficients may be solved by symbolic means.

The connection relation (3.21) was given by Sánchez-Ruiz et al. [35] (via a general expansion formula first derived by Verma [32]) to state an expansion formula relating the wave functions of two quantum systems described by Morse potentials of different parameters.

Table 1
Inversion coefficients for classical continuous polynomial sets, $x^n = \sum_{m=0}^{n} I_m(n) P_m(x)$

Polynomials	Polynomial sets, $\{P_n\}_{n \geqslant 0}$	Inversion coefficients, $I_m(n)$
Hermite	$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$	$\begin{cases} \frac{n!}{2^n k!(n-2k)!} & \text{if } m = n-2k \\ 0 & \text{otherwise} \end{cases}$
Laguerre	$e^t {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; -xt\right) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n} t^n$	$(-n)_m \frac{(\alpha+1)_n}{(\alpha+1)_m}$
Gegenbauer	$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x) t^n$	$\begin{cases} \frac{(n-2k+\lambda)n!}{k!(\lambda)_{n-k+1} 2^n} & \text{if } m = n-2k \\ 0 & \text{otherwise} \end{cases}$
Shifted Jacobi	$(1-t)^{-\lambda} {}_2F_1\left(\begin{matrix} \Delta(\lambda, 2) \\ \alpha+1 \end{matrix}; \frac{-2xt}{(1-t)^2}\right)$ $= \sum_{n=0}^{\infty} \frac{(\lambda)_n P_n^{(\alpha,\beta)}(1-x)}{(1+\alpha)_n} t^n,\ \lambda = \alpha+\beta+1$	$(-n)_m 2^n \frac{(2m+\lambda)}{(\lambda+m)_{n+1}} \frac{(\alpha+1)_n}{(\alpha+1)_m}$

Table 2
Inversion coefficients for classical discrete polynomial sets, $(-x)_n = \sum_{m=0}^{n} I_m(n) P_m(x)$

Polynomials	Polynomial sets, $\{P_n\}_{n \geqslant 0}$	Inversion coefficients, $I_m(n)$
Charlier	$c_n^a(x) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix}; -\frac{1}{a}\right)$	$\binom{n}{m}(-1)^{n-m} a^n$
Meixner	$M_n(x; \beta, c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - \frac{1}{c}\right)$	$\binom{n}{m}(-1)^m (\beta)_n \left(\frac{c}{c-1}\right)^n$
Krawtchouk	$K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p}\right)$	$\binom{n}{m}(-1)^m (-N)_n p^n$
Hahn	$Q_n(x; \alpha, \beta, N)$ $= {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}; 1\right)$	$(-1)^m \binom{n}{m} \frac{2m+\alpha+\beta+1}{(\alpha+\beta+1+m)_{n+1}} (\alpha+1)_n (-N)_n$

4. Connection and inversion coefficients for classical orthogonal polynomials

In this section, we give the inversion and connection coefficients for classical continuous and discrete orthogonal polynomials without using the orthogonality conditions. That follows from the results obtained in Section 3. For example, the inversion and connection coefficients for Hahn polynomials follow respectively from (3.9) and (3.10), the remainder formulas in Table 3 were obtained either as limit cases [12] or by means of (3.1) and (3.2).

Let us mention here that the inversion and connection coefficients between all classical orthogonal polynomials are given in the literature via other ways. For instance, Gasper [9] and Dehesa et al. [26,28,34] used the orthogonality conditions, Rainville [18] rearranged some series, and Ronveaux et al. [21,25] solved this question by the so-called `navima`-algorithm. Notice also that the inversion coefficients for Jacobi and shifted Jacobi polynomials were already obtained in [14, p. 277].

Table 3
Connection coefficients between classical discrete orthogonal polynomials, $Q_n(x)=\sum_{m=0}^{n} C_m(n)P_m(x)$

$P_m(x)$ \ $Q_n(x)$	Hahn $Q_m(x;\alpha,\beta,N)$; $\lambda=\alpha+\beta+1$	Meixner $M_m(x;\beta,c)$	Krawtchouk $K_m(x;p,N)$	Charlier $c_m^a(x)$
Hahn $Q_n(x;\gamma,\delta,M)$; $\mu=\gamma+\delta+1$	$\binom{n}{m}\frac{(\mu+n)_m(\alpha+1)_m(-N)_m}{(\lambda+m)_m(\gamma+1)_m(-M)_m}$ $\times\,{}_4F_3\left(\begin{matrix}m-n,m+n+\mu,\alpha+m+1,m-N\\ 2m+\lambda+1,\gamma+m+1,m-M\end{matrix};1\right)$	$\binom{n}{m}\frac{(\mu+n)_m(\beta)_m}{(\gamma+1)_m(-M)_m}\left(\frac{c}{c-1}\right)^m$ $\times\,{}_3F_2\left(\begin{matrix}m-n,m+n+\mu,m+\beta\\ \gamma+m+1,m-M\end{matrix};\frac{c}{c-1}\right)$	$\binom{n}{m}\frac{(\mu+n)_m(-N)_m}{(\gamma+1)_m(-M)_m}p^m$ $\times\,{}_3F_2\left(\begin{matrix}m-n,m+n+\mu,m-N\\ \gamma+m+1,m-M\end{matrix};p\right)$	$\binom{n}{m}\frac{(\mu+n)_m(-a)^m}{(\gamma+1)_m(-M)_m}$ $\times\,{}_2F_2\left(\begin{matrix}m-n,m+n+\mu\\ \gamma+m+1,m-M\end{matrix};-a\right)$
Meixner $M_n(x;\gamma,d)$	$\binom{n}{m}\frac{(\alpha+1)_m(-N)_m}{(\lambda+m)_m(\gamma)_m}\left(\frac{d-1}{d}\right)^m$ $\times\,{}_3F_2\left(\begin{matrix}m-n,\alpha+m+1,m-N\\ 2m+\lambda+1,\gamma+m\end{matrix};\frac{d-1}{d}\right)$	$\binom{n}{m}\frac{(\beta)_m}{(\gamma)_m}\left(\frac{c(1-d)}{(1-c)d}\right)^m$ $\times\,{}_2F_1\left(\begin{matrix}m-n,m+\beta\\ \gamma+m\end{matrix};\frac{c(1-d)}{(1-c)d}\right)$	$\binom{n}{m}\frac{(-N)_m}{(\gamma)_m}\left(\frac{p(d-1)}{d}\right)^m$ $\times\,{}_2F_1\left(\begin{matrix}m-n,m-N\\ \gamma+m\end{matrix};\frac{p(d-1)}{d}\right)$	$\binom{n}{m}\frac{1}{(\gamma)_m}\left(\frac{a(1-d)}{d}\right)^m$ $\times\,{}_1F_1\left(\begin{matrix}m-n\\ \gamma+m\end{matrix};\frac{a(1-d)}{d}\right)$
Krawtchouk $K_n(x;q,M)$	$\binom{n}{m}\frac{(\alpha+1)_m(-N)_m}{(\lambda+m)_m(-M)_m}q^{-m}$ $\times\,{}_3F_2\left(\begin{matrix}m-n,\alpha+m+1,m-N\\ 2m+\lambda+1,m-M\end{matrix};\frac{1}{q}\right)$	$\binom{n}{m}\left(\frac{c}{q(c-1)}\right)^m\frac{(\beta)_m}{(-M)_m}$ $\times\,{}_2F_1\left(\begin{matrix}m-n,\alpha+m\\ m-M\end{matrix};\frac{c}{q(c-1)}\right)$	$\binom{n}{m}\left(\frac{p}{q}\right)^m\frac{(-N)_m}{(-M)_m}$ $\times\,{}_2F_1\left(\begin{matrix}m-n,m-N\\ m-M\end{matrix};\frac{p}{q}\right)$	$\binom{n}{m}\left(-\frac{a}{q}\right)^m\frac{1}{(-M)_m}$ $\times\,{}_1F_1\left(\begin{matrix}m-n\\ m-M\end{matrix};-\frac{a}{q}\right)$
Charlier $c_n^b(x)$	$\binom{n}{m}\frac{(\alpha+1)_m(-N)_m}{(\lambda+m)_m(-b)^m}$ $\times\,{}_3F_1\left(\begin{matrix}m-n,\alpha+m+1,m-N\\ 2m+\lambda+1\end{matrix};-\frac{1}{b}\right)$	$\binom{n}{m}\left(\frac{c}{b(1-c)}\right)^m(\beta)_m$ $\times\,{}_2F_0\left(\begin{matrix}m-n,\beta+m\\ -\end{matrix};\frac{c}{b(1-c)}\right)$	$\binom{n}{m}\left(-\frac{p}{b}\right)^m(-N)_m$ $\times\,{}_2F_0\left(\begin{matrix}m-n,m-N\\ -\end{matrix};-\frac{p}{b}\right)$	$\binom{n}{m}\frac{a^m}{b^n}(b-a)^{n-m}$

Table 4
Connection coefficients between continuous classical orthogonal polynomials, $Q_n(x)=\sum_{m=0}^{n} C_m(n)P_m(x)$

$Q_n(x)$ \ $P_m(x)$	Shifted Jacobi $P_m^{(\alpha,\beta)}(1-x);\ \lambda=\alpha+\beta+1$	Gegenbauer $G_m^{\nu}(x)$	Laguerre $L_m^{(a)}(x)$	Hermite $H_m(x)$
Shifted Jacobi $P_n^{(\gamma,\delta)}(1-x)$, $\mu=\gamma+\delta+1$	$\frac{1}{(n-m)!}\frac{(\gamma+1)_n(\mu+n)_m}{(\gamma+1)_m(\lambda+m)_m}\times{}_3F_2\left(\begin{matrix}m-n,\ m+n+\mu,\ \alpha+m+1\\ \gamma+m+1,\ 2m+\lambda+1\end{matrix};1\right)$	$\frac{(\gamma+1)_n(-n)_m(\mu+n)_m}{(\gamma+1)_m(\nu)_m4^m n!}\times{}_4F_3\left(\begin{matrix}\Delta(2,m-n),\Delta(2,\mu+n+m)\\ \Delta(2,\gamma+m+1),\nu+m+1\end{matrix};\frac{1}{4}\right)$	$\frac{(\gamma+1)_n(\mu+n)_m}{(\gamma+1)_m(n-m)!2^m}\times{}_3F_1\left(\begin{matrix}m-n,m+n+\mu,a+m+1\\ \gamma+m+1\end{matrix};\frac{1}{2}\right)$	$\frac{(\gamma+1)_n(-n)_m(\mu+n)_m}{(\gamma+1)_m n!m!2^{2m}}\times{}_4F_2\left(\begin{matrix}\Delta(2,m-n),\Delta(2,\mu+n+m)\\ \Delta(2,\gamma+m+1)\end{matrix};\frac{1}{4}\right)$
Gegenbauer $G_n^{\omega}(x)$	$\frac{(-n)_m4^n(2m+\lambda)(\omega)_n(\alpha+1)_n}{n!(\lambda+m)_{n+1}(\alpha+1)_m}\times{}_4F_3\left(\begin{matrix}\Delta(2,m-n),\Delta(2,-\lambda-n-m)\\ \Delta(2,-\alpha-n),1-\omega-n\end{matrix};\frac{1}{4}\right)$	$\frac{(\nu+n-2k)(\omega-\nu)_k(\omega)_{n-k}}{k!(\nu)_{n+1-k}}\delta_{m,n-2k}$	$\frac{(-n)_m2^n(a+1)_n(\omega)_n}{(a+1)_m n!}\times{}_2F_3\left(\begin{matrix}\Delta(2,m-n)\\ \Delta(2,-a-n),1-\omega-n\end{matrix};\frac{1}{4}\right)$	$\frac{(-1)^k(\omega)_{n-k}}{k!(n-2k)!}\times{}_2F_0\left(\begin{matrix}-k,\omega+n-k\\ -\end{matrix};1\right)\delta_{m,n-2k}$
Laguerre $L_n^{(b)}(x)$	$\frac{(b+1)_n(2m+\lambda)2^m}{(b+1)_m(n-m)!(\lambda+m)_{m+1}}\times{}_2F_2\left(\begin{matrix}m-n,m+\alpha+1\\ 2m+\lambda+1,b+m+1\end{matrix};2\right)$	$\frac{(b+1)_n(-n)_m}{(b+1)_m n!2^m(\nu)_m}\times{}_2F_3\left(\begin{matrix}\Delta(2,m-n)\\ \Delta(2,b+m+1),\nu+m+1\end{matrix};\frac{1}{4}\right)$	$\frac{(b-a)_{n-m}}{(n-m)!}$	$\frac{(b+1)_n(-n)_m}{(b+1)_m m!n!2^m}\times{}_2F_2\left(\begin{matrix}\Delta(2,m-n)\\ \Delta(2,b+m+1)\end{matrix};\frac{1}{4}\right)$
Hermite $H_n(x)$	$\frac{(-n)_m4^n(2m+\lambda)(\alpha+1)_n}{(\alpha+1)_m(\lambda+m)_{n+1}}\times{}_4F_2\left(\begin{matrix}\Delta(2,m-n),\Delta(2,-\lambda-n-m)\\ \Delta(2,-\alpha-n)\end{matrix};\frac{1}{4}\right)$	$\frac{(-1)^k n!(n-2k+\nu)}{k!(\nu)_{n-2k+1}}\times{}_1F_1\left(\begin{matrix}-k\\ \nu-2k+n+1\end{matrix};1\right)\delta_{m,n-2k}$	$\frac{(a+1)_n(-n)_m2^n}{(a+1)_m}\times{}_2F_2\left(\begin{matrix}\Delta(2,m-n)\\ \Delta(2,-a-n)\end{matrix};-\frac{1}{4}\right)$	$\delta_{n,m}$

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